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COMMENT

Comment on 'Cyclic rotations, contractibility and Gauss–Bonnet'

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Abstract
It is shown that the proof of the Hannay formula, given by him, is based on an improper argument. An improved consideration is suggested.

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(Some figures in this article are in colour only in the electronic version)

1. Introduction

In [1], Hannay considers an open narrow ribbon with identical orientation at its ends. The ribbon may be thought of as a 'trace' of a rigid frame freely moving in space with the unit velocity $\boldsymbol{u}(s)$ and the angular velocity $\boldsymbol{\omega}(s)$. Hannay found a new interesting relation

$$2\pi n = \Omega + \int \boldsymbol{\omega} \cdot \boldsymbol{u}\, \mathrm{d}s \mod 4\pi \tag{1}$$

between the contractibility[2] number n, the solid angle Ω enclosed by the vector $\boldsymbol{u}(s)$, and the twist of the ribbon $\mathcal{T}w$ which is the integral in (1) divided by 2π. The contractibility number n is zero when the sequence of orientations of the frame is contractible and unity otherwise [3].

Equation (1) follows from the Călugăreanu–White–Fuller decomposition of the linking number [4–6]

$$\mathcal{L}k = \mathcal{T}w + \mathcal{W}r \tag{2}$$

and the Fuller theorem [7] that relates the writhe $\mathcal{W}r$ and the solid angle Ω

$$\mathcal{W}r = 1 + \Omega/2\pi \mod 2 \tag{3}$$

applied to the ribbon closed in a specific way.

The construction of the closure is a central point in the proof of the Hannay relation (1).

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[2] Hannay calls this quantity the 'turn number', but that name may be confused with the 'turning number' defined for a plane (closed) curve [2].

2. Following Hannay

Now I reformulate the proof given by Hannay a little more formally.

Consider a non-self-intersecting curve $A = \boldsymbol{r}(s) : [0, L] \to \mathbb{R}^3$ of class C^3, s being the arclength. Let $\boldsymbol{u}(s) = \frac{\mathrm{d}\boldsymbol{r}}{\mathrm{d}s}$, $\|\boldsymbol{u}(s)\| = 1$, and choose the unit vector $\boldsymbol{v}(s) \in C$ such that $\boldsymbol{u}(s) \cdot \boldsymbol{v}(s) = 0, \forall s \in [0, L]$. Let $\epsilon > 0$ be small enough so that the ribbon $R = \{\boldsymbol{r} + \mu\boldsymbol{v}, -\epsilon \leqslant \mu \leqslant \epsilon\}$ does not cross itself. Also it requires that $\boldsymbol{u}(0) = \boldsymbol{u}(L)$ and $\boldsymbol{v}(0) = \boldsymbol{v}(L)$.

According to [1], such a ribbon may be closed by adding a new one, Q, based on the curve $\boldsymbol{\rho}(\sigma), \sigma \in [0, l], \sigma$ the arclength: $Q = \{\boldsymbol{\rho} + \mu\boldsymbol{w}, -\epsilon \leqslant \mu \leqslant \epsilon\}$, where $\boldsymbol{w}(\sigma) \in C$ is a unit vector so that $\boldsymbol{\rho}'(\sigma) \cdot \boldsymbol{w}(\sigma) = 0, \forall\sigma \in [0, l]$, the prime denotes the derivative with respect to σ. The closure conditions may be written as

$$\boldsymbol{\rho}(0) = \boldsymbol{r}(L), \quad \boldsymbol{\rho}(l) = \boldsymbol{r}(0) \qquad \text{and} \qquad \boldsymbol{w}(0) = \boldsymbol{v}(L) = \boldsymbol{v}(0) = \boldsymbol{w}(l).$$

Following Hannay, it requires that Q lie in the plane G spanned by the vectors $\boldsymbol{v}(0), \boldsymbol{r}(L) - \boldsymbol{r}(0)$, i.e. $\boldsymbol{\rho}(\sigma) \in G, \boldsymbol{w}(\sigma) \in G, \forall\sigma \in [0, l]$. Moreover, Hannay requires that $\boldsymbol{\rho}'(0) = \boldsymbol{\rho}'(l)$. Since, in general, $\boldsymbol{u}(0) = \boldsymbol{u}(L) \notin G$, the closed ribbon $R \cup Q$ is based on a curve which may be non-smooth in the points where both parts are glued together. In other words, the tangent may jump for $s = 0, L$. The tangent indicatrix for such a curve consists of two parts: (1) a closed curve corresponding to the outward curve $\boldsymbol{r}(s)$ (it confines an area Ω on the surface of a unit sphere) and (2) either an arc of a great circle passed forth and back or the entire great circle. Note that these two parts may or may not have common points.

The key point of the proof given in [1] is an application of the Fuller formula for writhe (3) to this closed curve. In turn, the Fuller theorem is based on the Gauss–Bonnet and the Călugăreanu–White–Fuller theorems. The application of both of them to the case under consideration allows us to obtain the expression for the writhe

$$\mathcal{W}r = \Omega/2\pi \mod 1. \tag{4}$$

However, Hannay's claim is that (4) is valid modulo 2. It is easy to see that (4) does not hold modulo 2 in a general case. Let us take just an example. Consider an outward curve $\boldsymbol{r}(s)$ that is already closed, i.e. let $\boldsymbol{r}(0) = \boldsymbol{r}(L)$ (in addition to $\boldsymbol{u}(0) = \boldsymbol{u}(L)$). Then there is no need to build the closing ribbon at all. The Fuller formula may be applied to $\boldsymbol{r}(s)$ only and it yields

$$\mathcal{W}r = 1 + \Omega/2\pi \mod 2 \tag{5}$$

which is in accordance with (4), but contradicts Hannay's claim: $\mathcal{W}r = \Omega/2\pi \mod 2$.

Figure 1 presents another example when the Fuller theorem in Hannay's interpretation, having been applied to a non-smooth ribbon, leads to a wrong conclusion. The outward section R is based on the plane curve $y = y_0 = \text{const}$, $z = \sin x$, $x \in [0, 2\pi]$, $\boldsymbol{v} = \mathbf{const}$ and is directed along the y-axis. The flat return section Q is built as described in [1]. It is generated by the plane curve $y = y_0 + \sin^2\frac{x}{2}$, $z = 0$, $x \in [0, 2\pi]$, $\boldsymbol{w}$ lies in the xy-plane. Both parts are not twisted and the linking number of the closed ribbon $R \cup Q$ is $\mathcal{L}k = 1$ (it can be readily computed using the projection shown in the figure). Equation (2) suggests that $\mathcal{W}r_{R\cup Q} = 1$. However, the tangent indicatrix consists of two great circle arcs which means that the area enclosed is zero. Hence, $\mathcal{W}r_{R\cup Q} \neq \Omega/2\pi \mod 2$.

According to [8], (3) is valid for a curve for which a continuous deformation (with continuous tangents) to a circle exists. The continuity of tangents leads to the continuity of area. Thus, the last is sufficient in order that the Fuller theorem (3) might be applied. For the ribbon $R \cup Q$ in figure 1, this is actually the case, as can be seen from figure 2, where R is deformed in such a way that the tangent becomes continuous. (The new band fragment is based on the curve which is a union of three parts: (1) $x = -\pi \sin\frac{9}{4}t$, $z = 1 - \cos\frac{9}{4}t$, $t \in \left[0, \frac{2}{3}\pi\right)$;

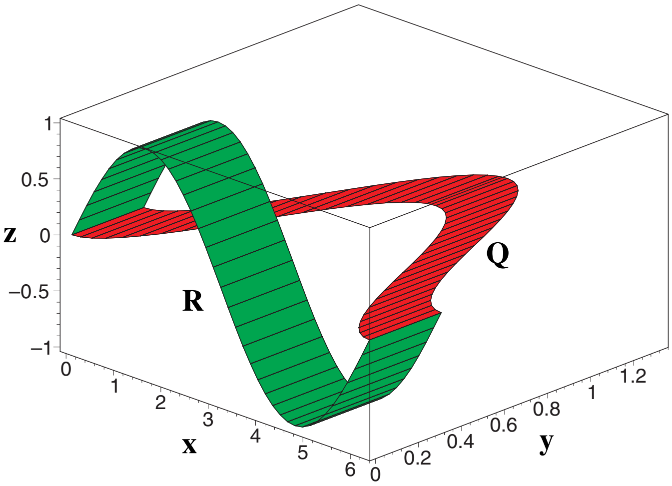

Figure 1. An example to show that Hannay's interpretation of the Fuller formula (3) does not work for a non-smooth ribbon.

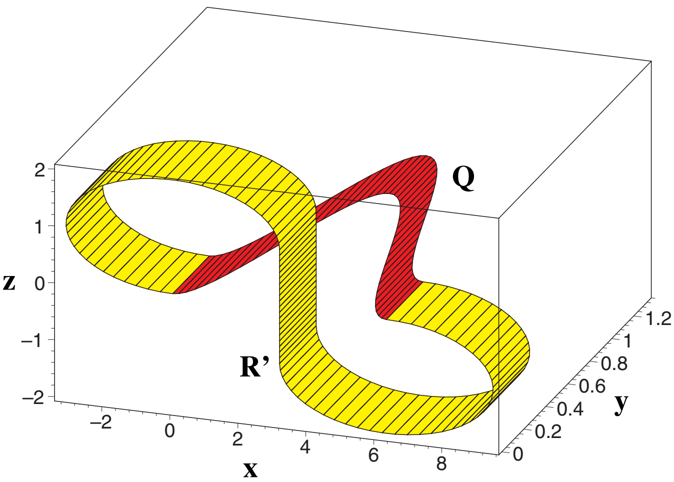

Figure 2. The outward ribbon R' is modified to make the tangent continuous.

(2) $x = \pi$, $z = 3\left(1 - \frac{t}{\pi}\right)$, $t \in \left[\frac{2}{3}\pi, \frac{4}{3}\pi\right)$; (3) $x = 2\pi + \pi \cos \frac{9}{4}t$, $z = \sin \frac{9}{4}t - 1$, $t \in \left[\frac{4}{3}\pi, 2\pi\right]$; $y = y_0$. Of course, it is easy to modify the curve locally to make it of class C^3.) Neither the total area changes by the deformation $R \to R'$, nor the writhe and now it is evident that the basic curve in figure 2 may be smoothly deformed to a circle. Since the total area swept out by the tangent indicatrix, is zero, the Fuller theorem implies $1 + \mathcal{W}r = 0 \bmod 2$ which is consistent with $\mathcal{W}r = 1$.

An additional point to emphasize is that, contrary to Hannay's claim, the writhe depends on the shape of the closure. Consider a different closing ribbon Q' of the same outward ribbon R as in figure 1. Q' lies in the same plane as Q does and it is generated by the curve consisting of three fragments: (1) $x = -2\pi \sin \frac{3}{2}t$, $y = y_0 + 1 - \cos \frac{3}{2}t$, $t \in \left[0, \frac{2}{3}\pi\right)$;

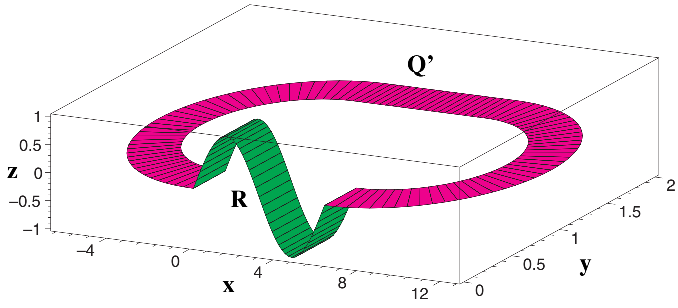

Figure 3. The outward ribbon R is the same as in figure 1, but the plane closure Q' changed.

(2) $x = 3t - 2\pi$, $y = y_0 + 2$, $t \in \left[\frac{2}{3}\pi, \frac{4}{3}\pi\right)$; (3) $x = 2\pi + 2\pi \sin \frac{3}{2}t$, $y = y_0 + 1 + \cos \frac{3}{2}t$, $t \in \left[\frac{4}{3}\pi, 2\pi\right]$; $z = 0$ (figure 3).

The twist of Q' is zero by definition and it can be easily seen that the linking number of $R \cup Q'$ is zero, as well. The Călugăreanu–White–Fuller formula implies $\mathcal{W}r_{R\cup Q'} = 0$. The area swept by the tangent indicatrix is 2π in this case and (3) is satisfied. Thus, this example shows that the writhe depends on the shape of the closing ribbon even if the last is formed in compliance with the limitations of [1].

Hannay also found an expression that relates the linking number of the closed ribbon to the contractibility number of the outward section: $\mathcal{L}k \bmod 2 = n$, which is not generally valid either. It will be clear from the following.

3. The proof of the Hannay formula

In this section, a corrected proof of (1) will be given.

In order to avoid the difficulties with a non-continuous tangent, a different closing ribbon P will be constructed. It is based on an oriented curve of class C^1 lying in the oriented plane H spanned by the vectors $\boldsymbol{u}(0)$, $\boldsymbol{r}(L) - \boldsymbol{r}(0)$. (If $\boldsymbol{u}(0)$, $\boldsymbol{r}(L) - \boldsymbol{r}(0)$ are parallel and therefore do not define a unique plane, the plane $H = \mathrm{Span}\{\boldsymbol{u}(0), \boldsymbol{v}(0)\}$ is taken.) This curve may be always chosen having neither self-intersections nor intersections with the outward part. Denote this curve by $\gamma(t)$, $t \in [0, \lambda]$, t being the arclength. It requires that $\gamma(0) = \boldsymbol{r}(L)$ and $\gamma(\lambda) = \boldsymbol{r}(0)$ as well as $\gamma'(0) = \boldsymbol{u}(L)$ and $\gamma'(\lambda) = \boldsymbol{u}(0)$, here the prime stands for the derivative with respect to t; thus the continuity of the tangent to the whole closed curve $\boldsymbol{r} \cup \gamma$ is provided. The closing ribbon P is based on γ and, if $\boldsymbol{v}(0) \in H$, then it may be chosen lying entirely in the same plane. Obviously its twist is zero then.

In the case when $\boldsymbol{v}(0) \notin H$, the ribbon P is generated by a constant vector $\boldsymbol{v}(0) = \boldsymbol{v}(L)$. Note that $\gamma'(t) \cdot \boldsymbol{v}(0) \neq 0$ in general. There are various ways to see that the twist integral of P vanishes. The following consideration seems to be the easiest: let $\boldsymbol{\nu}$ be the normal to H. Consider another closed ribbon K based on γ and its planar closure (the closure of the closure!) and generated by $\boldsymbol{\nu}$. It should have no self-intersections and its $\mathcal{L}k_K = \mathcal{W}r_K = \mathcal{T}w_K = 0$. Now introduce a discontinuous ribbon K' based on the same planar closed curve as K. K' coincides with P for the γ part and K' is identical to K for the rest. K' may have two points of discontinuity at $\gamma(0)$ and $\gamma(\lambda)$. In these points the angles between the generating vectors (i.e. between $\boldsymbol{v}(0)$ and $\boldsymbol{\nu}$) are equal in value but of opposite sign (say, α and $-\alpha$). K' can be locally modified to make it continuous (the same way as this is done in [8], p 359). Then the twist of K' is $\mathcal{T}w_{K'} = \alpha + \mathcal{T}w_P - \alpha + 0 = 0$ (because $\mathcal{T}w_{K'} = \mathcal{L}k_{K'} - \mathcal{W}r_{K'}$ due to (2), and $\mathcal{L}k_{K'} = \mathcal{W}r_{K'} = 0$). The above reasoning involves consideration of a discontinuous

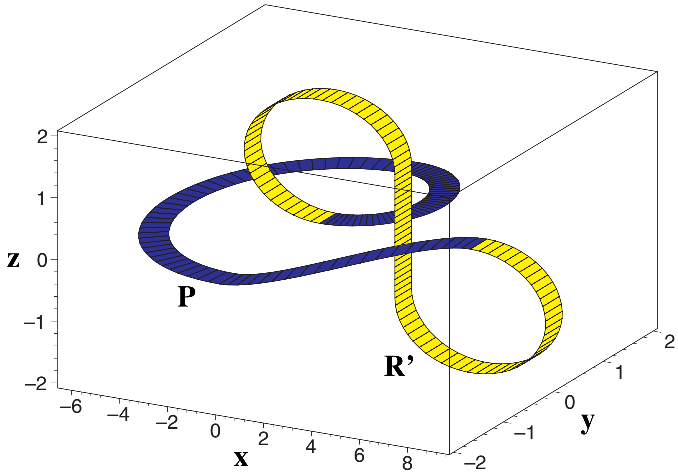

Figure 4. A different way to close the outward ribbon R' (see figure 2) keeping the tangent continuous.

ribbon which is related to the notion of a cord introduced by Fuller [7]. Torsionally-misaligned ribbons are also dealt with in [9] where a result was proved which states that the sum of the writhe and the twist of a discontinuous ribbon is equal to the linking number of the nearest closed one plus the normalized discontinuity angle. Application of this proposition to the non-modified ribbon K' leads to the same conclusion of zero twist of P.

Consider the tangent indicatrix of the closed curve $\boldsymbol{r} \cup \gamma$. Two possibilities should be taken into account.

1. The closure γ adds a zero-area appendage to the figure confined by the indicatrix of the outward curve $\boldsymbol{r}$, i.e. the closure indicatrix consists of an arc of a great circle (in particular, all the points of the great circle may belong to the indicatrix). It is essential that each point of the arc is counted an even number of times in both directions. The fragment Q may be considered as a particular example of such a closure for the outward section R' (figure 2). One can easily see that in this case no additional term comes, due to the appendage, into the Fuller formula for writhe and
$$\mathcal{W}r = 1 + \Omega/2\pi \quad \mod 2, \tag{6}$$
here Ω is the spherical area swept out by the tangent indicatrix of the outward curve.
2. The closure γ results in an entire great circle, each fragment of which is passed by an odd number of times. Figure 4 shows one possible ribbon P that has such a property. (The ribbon P displayed consists of three fragments: (1) $x = \pi \sin \frac{3}{2}t$, $y = y_0 + 1 - \cos \frac{3}{2}t$, $t \in \left[0, \frac{2}{3}\pi\right)$; (2) $x = 2\pi \sin \frac{3}{2}t$, $y = y_0 - 2\cos \frac{3}{2}t$, $t \in \left[\frac{2}{3}\pi, \frac{4}{3}\pi\right)$; (3) $x = 3t - 4\pi$, $y = y_0 - 2\cos^2 \frac{3}{4}t$, $t \in \left[\frac{4}{3}\pi, 2\pi\right]$; $z = 0$.) Therefore, 2π are to be added to the area term in the Fuller formula which now involves only the spherical area Ω corresponding to the outward curve:
$$\mathcal{W}r = \Omega/2\pi \quad \mod 2. \tag{7}$$

For any oriented curve $\beta(\tau), \tau \in I, I = [0, \Lambda]$ of class C^1 (τ is the arclength parametrization) lying in an oriented Euclidean plane E, the turning number $\mathcal{T}n_\beta$ may be defined as
$$\mathcal{T}n_\beta = \frac{1}{2\pi}\int_0^\Lambda \kappa_\beta(\tau)\,\mathrm{d}\tau$$

where $\kappa_\beta(\tau)$ is the signed curvature of $\boldsymbol{\beta}$. The turning number tells how many times the tangent turns around, as the curve is passed over.

If the tangents at the ends of the curve have the same directions (i.e. if $\beta'(0) = \beta'(\Lambda)$), then the turning number is integral. It is the degree of the map $\beta'(\tau) : I \mapsto S^1$. In particular, this is the case for a closed curve [2].

Two curves $\beta_0(\tau)$ and $\beta_1(\tau)$ in the plane E with $\beta_0'(0) = \beta_1'(0)$, $\beta_0'(\Lambda) = \beta_1'(\Lambda)$ of class C^1 are said to be homotopic if there exists a homotopy $F \in C^0([0,1] \times I \mapsto E)$ such that $F_0 = \beta_0$, $F_1 = \beta_1$, $F_\theta'(0) = \beta_0'(0)$, $F_\theta'(\Lambda) = \beta_0'(\Lambda)$ and $F_\theta : \tau \mapsto F(\theta, \tau)$ is a C^1 immersion for all $\theta \in [0, 1]$.

Two homotopic curves have the same turning number. For closed curves this follows from the invariance of the degree under homotopy [2]. To show this for non-closed curves, consider a pair of homotopic curves $\boldsymbol{c}_1$ and $\boldsymbol{c}_2$ connecting the points A_1 and A_2 with B_1 and B_2, respectively, and sharing the same initial and terminal tangents $\boldsymbol{T}_0$ and $\boldsymbol{T}_1$. Then there exists the plane closing homotopic curves $\boldsymbol{d}_1$ and $\boldsymbol{d}_2$ of class C^1 so that (1) they connect the points B_1 and B_2 with A_1 and A_2, respectively, (2) they have the initial tangent $\boldsymbol{T}_1$ and terminal $\boldsymbol{T}_0$, (3) they have the same turning number $\mathcal{T}n_d$ (for certainty, let $0 \leqslant \mathcal{T}n_d < 1$, then $\mathcal{T}n_d$ equals an angle through which the vector $\boldsymbol{T}_1$ has to be rotated in the positive direction to make it coincide with $\boldsymbol{T}_0$). Due to the additivity property of the turning number integral, $\mathcal{T}n_{c1\cup d1} = \mathcal{T}n_{c1} + \mathcal{T}n_d$ and $\mathcal{T}n_{c2\cup d2} = \mathcal{T}n_{c2} + \mathcal{T}n_d$. But the closed curves $\boldsymbol{c}_1 \cup \boldsymbol{d}_1$ and $\boldsymbol{c}_2 \cup \boldsymbol{d}_2$ are homotopic, hence $\mathcal{T}n_{c1\cup d1} = \mathcal{T}n_{c2\cup d2}$, which means that $\mathcal{T}n_{c1} = \mathcal{T}n_{c2}$.

In other words, one can deform a curve by moving its ends in the plane and keeping tangential directions at the ends but not allowing for cusps. The turning number remains unchangeable then.

Coming back to the closing curve $\boldsymbol{\gamma}$, it is easy to see now that both (6) and (7) may be considered as particular cases of the general relation

$$\mathcal{W}r = 1 + \Omega/2\pi + \mathcal{T}n_\gamma \mod 2 \tag{8}$$

where $\mathcal{T}n_\gamma$ is the turning number of $\boldsymbol{\gamma}$.

Hannay suggested relating the linking number of the closed ribbon to the number n which is 0 if the path corresponding to orientations of the outward ribbon is contractible in the $SO(3)$ space and $n = 1$ otherwise.

In what follows the path in $SO(3)$ corresponding to a ribbon V will be denoted by $\tilde{V}$.

Note that the closure P treated as a sequence of orientations, also makes a closed path in $SO(3)$. Let n_P be the contractibility number of $\tilde{P}$. Then

$$n_P = \mathcal{T}n_\gamma \mod 2. \tag{9}$$

Now examine what the linking number of the ribbon $R \cup P$ would be. To this purpose, formulate

Proposition 1. *The linking number of a closed ribbon U is*

$$\mathcal{L}k_U = n_U + 1 \mod 2 \tag{10}$$

where n_U is the contractibility number of $\tilde{U}$.

Firstly, note that (10) remains invariant under the homotopic deformations of U even if the ribbon may pass through itself that causes the value of the linking to jump by 2 only. The contractibility number cannot change under such deformations.

Secondly, consider a deformation of U transforming it into a closed ribbon $A \cup B$ based on a plane closed non-self-intersecting curve $\boldsymbol{a} \cup \boldsymbol{b}$, respectively, where $\boldsymbol{a}$ is a straight-line fragment and $\mathcal{T}n_b = 1$. By the construction, part A may be twisted but not B. The last immediately implies $n_B = 1$. If the twist of A is zero, then $\mathcal{L}k_U \bmod 2 = \mathcal{L}k_{A\cup B} = 0$ and $n_U = n_{A\cup B} = n_B = \mathcal{T}n_b = 1$ because $\tilde{A}$ is simply a point.

Generally, path $\tilde{A}$ has the contractibility number $n_A = \mathcal{T}w_A \bmod 2$, where $\mathcal{T}w_A$ is the twist of A. It is easy to check that $n_{A\cup B} = n_A + n_B \bmod 2$. Then $n_U = \mathcal{T}w_A + 1 \bmod 2$ and, on the other hand, the Călugăeanu–White–Fuller formula implies $\mathcal{L}k_U = \mathcal{T}w_A \bmod 2$ that proves (10).

Corollary 2. *Let U be a closed ribbon so that two (different) points exist on it where its orientation is the same. (By orientation of the ribbon in a particular point, the directions both of the tangent to the basic curve and of the generating vector are meant.) Let U_1 and U_2 be the corresponding fragments of U so that $U = U_1 \cup U_2$. Then path $\tilde{U}$ has a self-intersection point (i.e. $\tilde{U}_1$ and $\tilde{U}_2$ each closed) and*

$$\mathcal{L}k_U = n_1 + n_2 + 1 \mod 2 \tag{11}$$

where n_1 and n_2 are the contractibility numbers of $\tilde{U}_1$ and $\tilde{U}_2$, respectively.

The contractibilty number of $\tilde{U}$ is $n = n_1 + n_2$ mod 2 because the fundamental group $\pi_1(SO(3)) = \mathbb{Z}_2$. Then (11) follows from proposition 1.

Turning back to the ribbon $R \cup P$, it is easy to see that it satisfies the conditions of corollary 2 with $U_1 = R$ and $U_2 = P$. Then, by (11)

$$\mathcal{L}k_{R\cup P} = n + n_P + 1 \mod 2. \tag{12}$$

Note that the last expression differs from that in [1] by the term $n_P + 1$. The path corresponding to P is contractible in $SO(3)$ for the first of the two possible types of the closure that were considered in the beginning of this section and it is not contractible in the second case.

Applying now the Călugăeanu–White–Fuller theorem to $R \cup P$ and making use of (8), (9) and (12) yield

$$n + \mathcal{T}n_\gamma + 1 = 1 + \Omega/2\pi + \mathcal{T}n_\gamma + \mathcal{T}w \mod 2$$

which, after cancelling $\mathcal{T}n_\gamma + 1$, is exactly Hannay's result.

4. Concluding remarks

Summing up, it is clear now that the shape of the closing ribbon affects the value of the writhe as well as the relationship between the linking number and the contractibility property, but the final equation remains unchanged because of the mutual compensation of the intermediate variations.

It is likely that the final result could be achieved by the careful consideration of the closure proposed by Hannay, but the above construction seems to make the whole proof more transparent. To compute the writhe for the closure with jumping tangents, one may apply the result that is formulated (but not proved) as conjecture 16 in [10].

Note also that the computation of the linking number of the closed ribbon may be done by pulling out the outward ribbon straight and by counting the number of twists in this fragment as was proposed in [1]. However, it seems that this approach is too intricate because of the necessity to account properly for the limiting shape of the closing section which could pass through itself or the outward section.

In some sense, the Hannay equation may be considered as an adaptation of the Fuller theorem (3) to a non-closed ribbon of a particular type. Equation (3) is based on the Gauss–Bonnet theorem, but, in distinction to the last, the Fuller formula, when applied to a non-smooth closed ribbon, includes no extra terms due to the non-continuity of the tangent or the principal normal. These non-area terms are taken up by the twist component.

Another view of the Hannay equation is to treat it as a refined Gauss–Bonnet theorem applied to an open ribbon with the closed tangent indicatrix. It is interesting to note that the Fuller formula stems from the classical Gauss–Bonnet, while, in a converse manner, the ‘generalized Gauss–Bonnet’ (i.e. the Hannay equation) follows from the Fuller theorem.

John Hannay authorizes me to declare here that he fully agrees that the principal one of his two proofs contains faulty reasoning, and the correction, in particular proposition 1, which he just missed, neatly clinches the proof.

Acknowledgments

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